Last Time: - Span and Subspaces

- Linear independence...

Def^{n}: Let V be a vector space. A set $S \subseteq V$ is linearly independent when for all $s_1, s_2, \dots, s_n \in S$ if

$$c_1 s_1 + c_2 s_2 + \cdots + c_n s_n = 0$$

then $c_1 = c_2 = \cdots = c_n = 0$.

NB: i.e. the only linear combination giving rise to 0_V is the "0 linear combination".

Remark: If $S = \{v_1, v_2, \dots, v_n\} \subseteq \mathbb{R}^d$ is finite, then S is lin. indep precisely when

$$[v_1 \mid v_2 \mid \cdots \mid v_n]\vec{x} = \vec{0}$$ has a unique solution.

columns

Ex: Decide if $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}$ is lin. indep.

Sol: We solve the system

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ Original system has the same solution set as:

$$\begin{cases} x \quad\; + z = 0 \\ y + 2z = 0 \\ \quad\;\; \uparrow \end{cases} \rightsquigarrow \begin{cases} x = -t \\ y = -2t \\ z = t \end{cases}$$

$\therefore$ Solution set is $\left\{ t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$.

As the system has infinitely many solutions, we have $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}$ is dependent! $\boxed{S}$

Ex: Is $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ lin indep?

Sol: We solve the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightsquigarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

$\therefore$ Solution set is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Hence the only lin comb of vectors in S to give zero vector is the 0 combination!

Hence $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ is lin. indep! :) $\boxed{//}$

Properties of Linear Independence

Prop: Let $S \subseteq V$ for some vector space V.

① If $A \subseteq S$ and S is lin. indep, then A is linearly independent.

② If $D \subseteq S$ and D is lin. dep., then S is linearly dependent.

pf: Let $S \subseteq V$ for vector space V.

①: Assume S is lin. indep and let $A \subseteq S$.

If A has a linear relationship

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0_V$$

for $v_1, v_2, \cdots, v_n \in A$, then $v_1, v_2, \cdots, v_n \in S$

Hence this is a linear combination of vectors in S. Because S is lin. indep, $c_1 = c_2 = \cdots = c_n = 0$.

Hence A is linearly indep by definition.

②: This is the contrapositive of ①. $\boxtimes$

Let $D \subseteq S$ and suppose D is lin. dep. Hence There are vectors $v_1, v_2, \cdots, v_n \in D$ and nonzero real numbers $c_1, c_2, \cdots, c_n \in \mathbb{R}$ such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0_V.$$

But $v_1, v_2, \cdots, v_n \in S$ because $D \subseteq S$, so this nonzero linear combination is also a combination of vectors in S. Hence S is linearly dependent. $\boxtimes$

Ex: Let V be a vector space. The empty set $\emptyset$ is linearly independent because it has no vectors to make a nonzero combination!

Prop Let $u \in S \subseteq V$ for some vector space V. Then [$u \in \text{Span}(S \setminus \{u\})$] if and only if [there is a nonzero linear dependence relation in S involving u.]

pf: Let $0_V \neq u \in S \subseteq V$ for vector space V.

($\Rightarrow$): Assume $u \in \text{Span}(S \setminus \{u\})$. Then u is a linear combination of vectors in $S \setminus \{u\}$.

Thus $u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ for some $v_1, v_2, \ldots, v_n \in S$ and $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Hence $0_V = (-1)u + c_1 v_1 + \cdots + c_n v_n$ is a nontrivial linear combination involving u.

($\Leftarrow$): Assume there is a linear dep relation in S involving u. Hence there are $a, c_1, c_2, \ldots, c_n \in \mathbb{R}$ with $a \neq 0$ and vectors $v_1, v_2, \ldots, v_n \in S \setminus \{u\}$ such that $0_V = au + c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$.

Thus $-au = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ holds by subtracting au from both sides. Now scalar multiply by $-\frac{1}{a}$ to obtain $u = \frac{-a}{-a} u = -\frac{1}{a}(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$,

and thus $u = \left(-\frac{c_1}{a}\right)v_1 + \left(-\frac{c_2}{a}\right)v_2 + \cdots + \left(-\frac{c_n}{a}\right)v_n$.

Hence $u \in \operatorname{Span}(S \setminus \{u\})$ as desired. $\boxtimes$

(A nontrivial linear combination is a linear combination of vectors with all involved scalars nonzero).

Remark on 0_V: Is $\{0_V\}$ lin indep? NO!

* $c0_V = 0_V$ for all $c \in \mathbb{R}$

so $1 0_V = 0_V$ is a nontrivial linear dep!

Hence $\{0_V\}$ is lin. dep. set! Hence any set containing 0_V is linearly dependent!

Cor: Let $S \subseteq V$ for vector space V. For all $u \in V \setminus S$ we have $u \in \operatorname{Span}(S)$ if and only if $S \cup \{u\}$ is linearly dependent.

union

Cor: For all $u \in V$ and all $S \subseteq V$ we have

$\operatorname{span}(S \cup \{u\}) = \operatorname{span}(S)$ if and only if $u \in \operatorname{Span}(S)$.

pf: Let $u \in V$ and $S \subseteq V$.

($\Rightarrow$): Suppose $\operatorname{span}(S \cup \{u\}) = \operatorname{span}(S)$. Note $u \in S \cup \{u\} \subseteq \operatorname{span}(S \cup \{u\}) = \operatorname{span}(S)$, so $u \in \operatorname{span}(S)$ as desired. $\boxtimes$

($\Leftarrow$): Suppose $u \in \operatorname{Span}(S)$. Thus $u = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ for some $v_1, \ldots, v_n \in S \setminus \{u\}$. $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Now any linear combination involving u can be rewritten using

$v_1, v_2, \cdots, v_n$. Hence, $\text{Span}(S \cup \{u\}) \subseteq \text{Span}(S)$.

Thus we have $\text{Span}(S \cup \{u\}) = \text{span}(S)$. $\boxtimes$

<u>Cor</u>: Let V be a vector space. Subset $S \subseteq V$ is linearly indep if and only if for all $u \in S$ we have $\text{span}(S \setminus \{u\}) \subsetneq \text{Span}(S)$.

<u>pf</u>: Let V be a vector space and $S \subseteq V$.

<u>$(\Rightarrow)$</u>: Suppose S is lin indep. Let $u \in S$ be arbitrary. Now $u \in \text{Span}(S)$. If $u \in \text{Span}(S \setminus \{u\})$, then there would be a linear dependence in $(S \setminus \{u\}) \cup \{u\} = S$ by the proposition! As S is lin. indep, $u \notin \text{Span}(S \setminus \{u\})$ so $\text{Span}(S \setminus \{u\}) \neq \text{Span}(S)$.

<u>$(\Leftarrow)$</u>: Suppose $\text{span}(S \setminus \{u\}) \subsetneq \text{span}(S)$ for all $u \in S$. Suppose S is lin. dep. Thus there is a nontrivial lin. dep. relation $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0_V$ for some vectors $v_1, v_2, \cdots, v_n \in S$ where $c_1, c_2, \cdots, c_n \in \mathbb{R}$ are all nonzero. Thus $c_1 \neq 0$. But this is a nontrivial linear dependence involving v_1, so $v_1 \in \text{Span}(S \setminus \{v_1\})$ by the proposition contradicting our assumption (b/c $\text{Span}(S \setminus \{v_1\}) = \text{Span}(S)$). Hence there is no nontrivial lin. dep. in S, so S is linearly independent. $\boxtimes$

Prop: Let V be a vector space. Every finite set $S \subseteq V$ has an $I \subseteq S$ such that

① I is lin. indep., and

② $\text{span}(I) = \text{span}(S)$.

pf: On hold... $\boxed{\diagup}$

Ex: Find a lin indep set contained in

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

with the same span.

Sol: Next time. $\boxed{\diagup}$